We have the following diagram of functors

$$\begin{tikzcd}
\mathrm{Lie}_k \arrow[rr, "\Psi"] \arrow[dr, bend right=20, "C^*_{\mathrm{Lie}}"'] & & \mathrm{FMP} \\
& (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \arrow[ur, "\mathrm{Spf}"'] \arrow[ul, bend left=40, "\mathfrak{D}"] \arrow[ul, phantom, "\perp"] &
\end{tikzcd}$$

Observe that

definition of Spf

$$\Psi(\mathfrak{g})(A) \simeq \mathrm{CAlg}_k^{\mathrm{aug}}\left(C^*_{\mathrm{Lie}}(\mathfrak{g}), A\right)$$

$$\simeq \mathrm{Lie}_k\left(\mathfrak{D}(A), \mathfrak{g}\right)$$

adjunction plus "opposites"

Hence Ψ preserves <u>limits</u> in Lie_k

$\Rightarrow$ Ψ admits a <u>left</u> adjoint Φ by adjoint functor theorem.

<u>small print</u>: need to show Ψ preserves filtered colimits, too

2

Recall that an adjunction $\Phi \dashv \Psi$ is an equivalence of ∞-categories if & only if

1. Ψ detects equivalences ($X \xrightarrow{f} Y$ $\simeq$ $\Leftrightarrow$ $\Psi(f)$ $\simeq$)

2. the unit $\eta : id_{FMP} \Rightarrow \Psi \circ \Phi$ is an equivalence

Tomorrow Claudia will discuss ②. Let me now prove ① using the following fact:

<u>Fact</u> For $k \oplus k[n]$ the n-shifted dual numbers,

$$\mathfrak{D}(k \oplus k[n]) \simeq \mathrm{Free}_{\mathrm{Lie}}(k[-1-n])$$

This should be plausible. since

$$C^*_{\mathrm{Lie}}(\mathrm{Free}_{\mathrm{Lie}}(k[-1-n])) \simeq k \oplus (k[-1-n])^\vee[-1]$$
$$\simeq k \oplus k[n]$$

We will spend much of today proving this fact and closely related things

Prop For $f: \mathfrak{g} \to \mathfrak{h}$ in Lie_k, if $\Psi(f)$ is an equivalence, then f is an equivalence.

Pf

$$\begin{tikzcd}
\Psi(\mathfrak{g})(k \oplus k[n]) \arrow[r, "\Psi(f)", "\simeq"'] \arrow[d, "\wr"] & \Psi(\mathfrak{h})(k \oplus k[n]) \arrow[d, "\wr"] \\
\mathrm{Lie}_k(\mathfrak{D}(k \oplus k[n]), \mathfrak{g}) \arrow[d, "\wr"] & \mathrm{Lie}_k(\mathfrak{D}(k \oplus k[n]), \mathfrak{h}) \arrow[d, "\wr"] \\
\mathrm{Lie}_k(\mathrm{Free}_{\mathrm{Lie}}(k[-1-n]), \mathfrak{g}) \arrow[d, "\wr"] & \mathrm{Lie}_k(\mathrm{Free}_{\mathrm{Lie}}(k[-1-n]), \mathfrak{h}) \arrow[d, "\wr"] \\
\mathrm{Mod}_k(k[-1-n], \Theta(\mathfrak{g})) & \mathrm{Mod}_k(k[-1-n], \Theta(\mathfrak{h}))
\end{tikzcd}$$

Running over all n, we see that $\Theta(f)$ is a quasi-isomorphism

$\square$

Understanding $\mathfrak{D}$

Our strategy, as earlier, is to study the linear shadow of $\mathfrak{D}$. Consider the composite

$$\begin{tikzcd} (\mathrm{CAlg}^{\mathrm{aug}}_k)^{op} \arrow[r, "\mathfrak{D}"] & \mathrm{Lie}_k \arrow[r, "\theta"] \arrow[l, bend left, "C^*_{\mathrm{Lie}}", "\top"'] & \mathrm{Mod}_k \arrow[l, bend left, "\mathrm{Free}_{\mathrm{Lie}}", "\top"'] \end{tikzcd}$$

We know there is a natural equivalence

$$C^*_{\mathrm{Lie}}(\mathrm{Free}_{\mathrm{Lie}}(V)) \simeq k \oplus V^\vee[-1],$$

that is

$$\begin{tikzcd} \mathrm{Mod}_k \arrow[r, "\mathbb{D}_k"] \arrow[rr, bend left, "C^*_{\mathrm{Lie}} \circ \mathrm{Free}_{\mathrm{Lie}}"] & \mathrm{Mod}_k^{op} \arrow[r, "\mathrm{SqExt}"] & (\mathrm{CAlg}^{\mathrm{aug}}_k)^{op} \\ V \arrow[r, mapsto] & \mathbb{D}_k(V) = V^\vee & \\ & M \arrow[r, mapsto] & k \oplus M[-1] \end{tikzcd}$$

square-zero extension

Now: ~~observe~~

1. $\mathbb{D}_k$ is left adjoint to itself:

$$\mathrm{Mod}_k(V, \mathbb{D}_k(W)) \simeq \mathrm{Mod}_k(V \otimes W, k) \simeq \mathrm{Mod}_k(W, \mathbb{D}_k(V))$$

$$\simeq \mathrm{Mod}_k^{op}(\mathbb{D}_k(V), W)$$

2. there is a <u>left adjoint</u> L to SqExt that we will now examine in some detail: it is a case of the cotangent complex

The cotangent complex

We will work at a convenient level of generality.

Fix k a characteristic zero field.

Let CAlg_k^{dg} denote the (model) category of

dg comm algebras over k.

For $A \in \mathrm{CAlg}_k^{dg}$, let CAlg_A^{dg} denote the

(model) category of comm dg algebras over A:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d] & A \arrow[d] \\ B \arrow[r] \arrow[ur, phantom, "\circlearrowright"] & B' \end{tikzcd}$$

morphism

objects

For $B \in \mathrm{CAlg}_A^{dg}$, let $(\mathrm{CAlg}_A^{dg})_{/B}$ denote the

(model) category of A-algebras sliced over B

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C' \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$$

morphism

objects

7

Then $\mathrm{SqExt} : \mathrm{Mod}^{dg}_B \longrightarrow \left(\mathrm{CAlg}^{dg}_A\right)_{/B}$

$$M \longmapsto \left(\begin{array}{c} B \oplus M \\ \downarrow{\scriptstyle \pi_B} \\ B \end{array}\right)$$

$$\begin{array}{c} B\oplus M \\ \| \\ B \ltimes M \end{array}$$

$(b,m)\cdot(b',m')$
$\|$
$(bb', bm' + b'm)$

is well-defined.

Moreover

universal property of Kähler differentials

$$\mathrm{CAlg}^{dg}_{/B}\left(C, B\ltimes M\right) \cong \mathrm{Der}^{dg}_A(C,M) \cong \mathrm{Mod}^{dg}_B\left(B\otimes_C \Omega^1_{C/A}, M\right)$$

$D_f \to \ker \pi = M$

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & B\ltimes M \arrow[d, "\pi_B"] \\ B \arrow[r, equal] & B \end{tikzcd} \longmapsto D_f = \pi_M \circ f$$

"indecomposables"

Recall: $\Omega^1_{C/A} = J/J^2$, $J = \mathrm{Ker}\left(C\otimes_A C \xrightarrow{m_C} C\right)$

base change from C to B

$$\mathrm{Mod}^{dg}_B\left(B\otimes_C \Omega^1_{C/A}, M\right) \cong \mathrm{Mod}^{dg}_C\left(\Omega^1_{C/A}, M\right) \ni g$$

$$\downarrow \cong \qquad\qquad \downarrow$$

$$\mathrm{Der}^{dg}_A(C,M) \ni g\circ d_{dR}$$

$d_{dR} : C \to \Omega^1_{C/A}$

$x \longmapsto [x\otimes 1 - 1\otimes x]$

Hence

check SqExt preserves fibrations & acyclic fibrations

<u>Prop</u> There is a Quillen adjunction

$$B \underset{-}{\otimes} \Omega^1_{-/A} : \left(\mathrm{CAlg}^{dg}_A\right)_{/B} \rightleftarrows \mathrm{Mod}^{dg}_B : \mathrm{SqExt}$$

<u>Def</u> The <u>cotangent complex</u> $\mathbb{L}_{B/A}$ is any B-module quasi-isomorphic to

$$B \underset{P}{\otimes} \Omega^1_{P/A}$$

where $P \xrightarrow{\simeq} B$ is a cofibrant replacement for B in CAlg^{dg}_A.

<u>Trivial example</u>

Let $A = k$, $B = k[x_1, \dots, x_n] = \mathrm{Sym}(\{x_1, \dots, x_n\})$

Then B is already cofibrant, since it's free

Hence $\mathbb{L}_{B/A} \simeq \Omega^1_{B/A} = \bigoplus_{i=1} k[x_1, \dots, x_n]\, dx_i$.

~~Similarly for $k[x_1, \dots, x_n]$~~

Rmk For more complicated B, one usually constructs a "semifree resolution"

$$P = \left(k[x_\alpha]_{\alpha \in I},\ \partial = \sum_\alpha f_\alpha \frac{\partial}{\partial x_\alpha} \right) \xrightarrow{\simeq} B$$

$$|f_\alpha| - |x_\alpha| = 1$$

&

$$f_\alpha \in k[x_\beta]_{\beta < \alpha}, \quad I \text{ ordered}$$

as $\Omega^1_{P/k}$ is easy to compute then. //

Lemma (Flat base change)

Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\text{P.O.}"] & A' \arrow[d] \\ B \arrow[r] & B' = A' \otimes_A B \end{tikzcd}$$

be a pushout diagram in $CAlg^{dg}_A$

where ϕ or f is flat. Then

$$B' \otimes_A \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{B'/A'}$$

is a quasi-iso of B'-modules.

Pf

Let $P \to B$ be a cofibrant replacement as an A-algebra.

Then

$$P' := A' \otimes_A P \longrightarrow A' \otimes_A B = B'$$

is a ~~weak~~ equivalence by the flatness assumption and P' is cofibrant as an A'-algebra.

Thus the natural map from base change

$$\begin{array}{ccc} B' \otimes_B \left(B \otimes_P \Omega^1_{P/A} \right) & \longrightarrow & B' \otimes_{P'} \Omega^1_{P'/A'} \\ \wr\| & & \wr\| \\ B' \otimes_B \mathbb{L}_{B/A} & & \mathbb{L}_{B'/A'} \end{array}$$

is a weak equivalence. $\square$

Similarly generalizing the right exact sequence in Kähler differentials:

Prop For $A \to B \to C$ maps in CAlg, the natural maps "transitivity"

$$C \otimes_B \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{C/A} \longrightarrow \mathbb{L}_{C/B}$$

form a cofiber sequence.

Application

$$
\mathrm{CAlg}_k^{\mathrm{aug}} = (\mathrm{CAlg}_k)_{/k} \underset{\mathrm{SqExt}}{\overset{L}{\rightleftarrows}} \mathrm{Mod}_k \qquad (\perp)
$$

w/ the shift!

where $L(A) \simeq \left(k \underset{A}{\otimes} \mathbb{L}_{A/k}\right)[-1]$

$\simeq \mathbb{L}_{k/A}$

use transitivity for $k \to A \to k$

Hence $\Theta \circ \mathfrak{D} \simeq \mathbb{D}_k \circ L : A \longmapsto \mathbb{L}_{k/A}^{\vee}$.

(Note that this is what you'd guess as a model for $\mathbb{T}_{k/A}$: the dual to cotangent!)

Now some useful results follow.

<u>Lemma</u>

For $\mathfrak{g} \in \mathrm{Lie}_k$ s.t.

(a) $\mathfrak{g}^d \equiv 0$, $d \leq 0$

(b) $\dim_k(\mathfrak{g}^d) < \infty$, $\forall d$,

the unit map $\mathfrak{g} \to \mathfrak{D}(C^*_{Lie}(\mathfrak{g}))$ is an equivalence.

STATE & PROVE COROLLARY FIRST!

Pf It suffices to verify there is an equivalence on the underlying cochain complexes.

The map $\Theta(\mathfrak{g}) \longrightarrow \Theta(\mathfrak{D}(C^*_{Lie}\mathfrak{g})) \simeq \mathbb{L}^\vee_{k/C^*_{Lie}(\mathfrak{g})}$

has a predual

$$\mathbb{L}_{C^*_{Lie}(\mathfrak{g})/k} \underset{C^*_{Lie}(\mathfrak{g})}{\otimes} k \longrightarrow \mathfrak{g}^\vee[-1].$$

We will prove <u>this</u> is an equivalence.

Recall that $C^*_{Lie}(\mathfrak{g})$ is $\prod_n \mathrm{Sym}^n(\mathfrak{g}^\vee[-1])$ as a graded algebra.
Fix a basis $x_1, \dots, x_n$ for $(\mathfrak{g}^{+1})^\vee$, so that

$$C^*_{Lie}(\mathfrak{g})^0 \cong k[[x_1, \dots, x_n]]$$

Let $A = \left(\bigoplus_n \mathrm{Sym}^n(\mathfrak{g}^\vee[-1]), \partial\right) \xhookrightarrow{\phi} C^*_{Lie}(\mathfrak{g})$

and so $A^0 = k[x_1 \dots x_n] \xhookrightarrow{\phi^0} k[[x_1 \dots x_n]]$

Note that by the hypotheses,

$$A \otimes_{k[x_1 \dots x_n]} k[[x_1 \dots x_n]] \xrightarrow{\cong} C^*_{Lie}(\mathfrak{g}) .$$

Indeed, the map

$$\begin{tikzcd} k[x_1 \dots x_n] \arrow[r] \arrow[d] & k[[x_1 \dots x_n]] \arrow[d] \\ A \arrow[r] & C^*_{Lie}(\mathfrak{g}) \end{tikzcd}$$

is a pushout square in CAlg_k as ϕ^0 is flat

and so $H^n(A) \otimes_{k[\dots]} k[[\dots]] \xrightarrow{\cong} H^n(\mathfrak{g})$.

Thus by flat base change:

$$\mathbb{L}_{C^*_{Lie}(g)/A} \underset{C^*_{Lie}(g)}{\otimes} k \simeq \mathbb{L}_{k[[x_1 \cdots x_n]]/k[x \cdots x_n]} \underset{k[[x \cdots x]]}{\otimes} k \simeq \mathbb{L}_{R/k} \simeq 0$$

where $R = k[[x \cdots x_n]] \underset{k[\cdots]}{\otimes} k \cong k$.

Now the predual map can be identified with

$$\mathbb{L}_{A/k} \underset{A}{\otimes} k \longrightarrow g^\vee[-1)$$

but A is cofibrant (as it is semifree) so

use cofiber sequence

$$\mathbb{L}_{A/k} \underset{A}{\otimes} k \simeq g^\vee[-1] . \qquad \square$$

Cor For $n \geq 0$, $\underset{\overset{\shortparallel}{g^{(n)}}}{\mathrm{Free}_{Lie}(k[-n-1])}$ satisfies the conditions (a) & (b) so

$$g^{(n)} \xrightarrow{\simeq} \mathfrak{D}(C^*_{Lie}(g^{(n)}))$$

so

$$\mathfrak{D}(k \oplus k(n)) \simeq g^{(n)} .$$