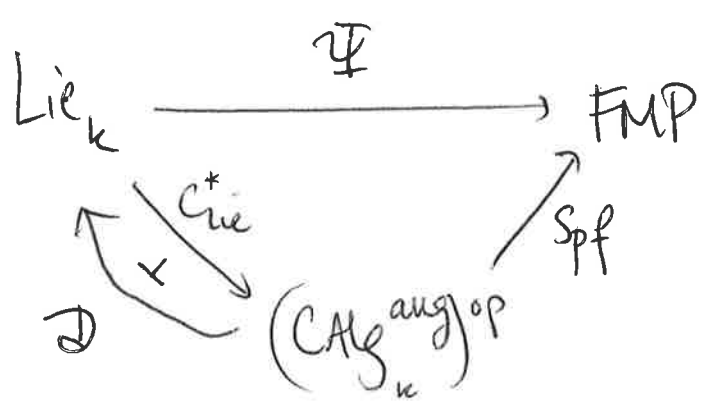


We have the following diagram of functors



Observe that

$$\Psi(g)(A) \cong \text{CAlg}_k^{\text{aug}}(C_{\text{Lie}}^*(g), A)$$

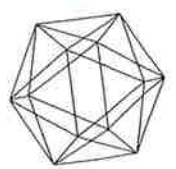
$$\cong \text{Lie}_k(\mathcal{D}(A), g)$$

adjunction plus "opposites"

Hence Ψ preserves limits in Lie_k

$\Rightarrow \Psi$ admits a left adjoint Φ by adjoint functor theorem.

small point: need to show Φ preserves filtered colimits, too



Recall that an adjunction $\Phi \dashv \Psi$ is an equivalence of ∞ -categories if & only if

① Ψ detects equivalences $(X \xrightarrow{f} Y \simeq \Leftrightarrow \Psi(f) \simeq)$

② the unit $\eta: \text{id}_{\text{FMP}} \Rightarrow \Psi \circ \Phi$ is an equivalence

Tomorrow Claudia will discuss ②. Let me now prove ① using the following fact:

Fact For $k \oplus k[n]$ the n -shifted dual numbers,

$$\mathcal{D}(k \oplus k[n]) \simeq \text{Free}_{\text{Lie}}(k[-1-n])$$

↪ This should be plausible since

$$\begin{aligned} C_{\text{Lie}}^*(\text{Free}_{\text{Lie}}(k[-1-n])) &\simeq k \oplus (k[-1-n])^\vee[-1] \\ &\simeq k \oplus k[n] \end{aligned}$$

We will spend much of today proving this fact and closely related things

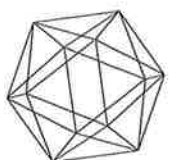


Prop For $f: g \rightarrow h$ in Lie_k , if $\Psi(f)$ is an equivalence, then f is an equivalence.

Pf

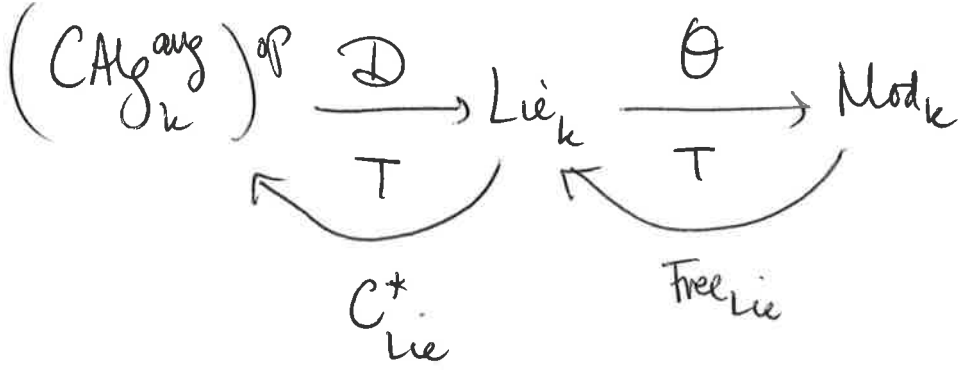
$$\begin{array}{ccc}
 & & \swarrow \Psi(f) \\
 \Psi(g)(k \oplus k[n]) & \cong & \Psi(h)(k \oplus k[n]) \\
 \cong & & \cong \\
 Lie_k(\mathcal{D}(k \oplus k[n]), g) & & Lie_k(\mathcal{D}(k \oplus k[n]), h) \\
 \cong & & \cong \\
 Lie_k(\text{Free}_{Lie}(k[-1-n]), g) & & Lie_k(\text{Free}_{Lie}(k[-1-n]), h) \\
 \cong & & \cong \\
 \text{Mod}_k(k[-1-n], \mathcal{O}(g)) & & \text{Mod}_k(k[-1-n], \mathcal{O}(h))
 \end{array}$$

Running over all n , we see that $\mathcal{O}(f)$ is a quasi-isomorphism □



Understanding \mathcal{D}

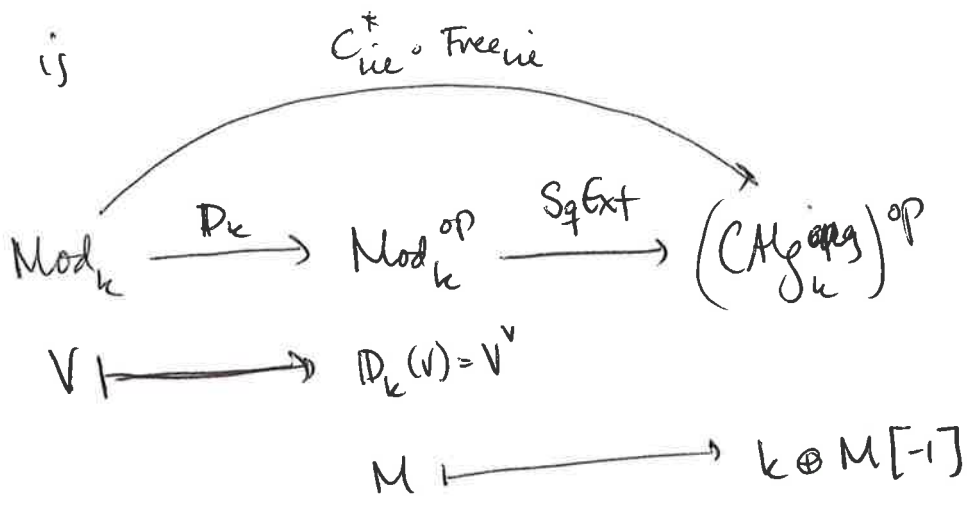
Our strategy, as earlier, is to study the linear shadow of \mathcal{D} \Rightarrow Consider the composite



We know there is a natural equivalence

$$C_{\text{Lie}}^*(\text{Free}_{\text{Lie}}(V)) \cong k \oplus V^{\vee}[-1],$$

that is



square-zero extension

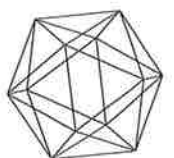


Now: ~~observe~~

① \mathbb{D}_k is left adjoint to itself:

$$\begin{aligned} \text{Mod}_k(V, \mathbb{D}_k(W)) &\cong \text{Mod}_k(V \otimes W, k) \cong \text{Mod}_k(W, \mathbb{D}_k(V)) \\ &\cong \text{Mod}_k^{\text{op}}(\mathbb{D}_k(V), W) \end{aligned}$$

② there is a left adjoint L to SqExt
 that we will now examine in some
 detail: it is a case of the cotangent complex



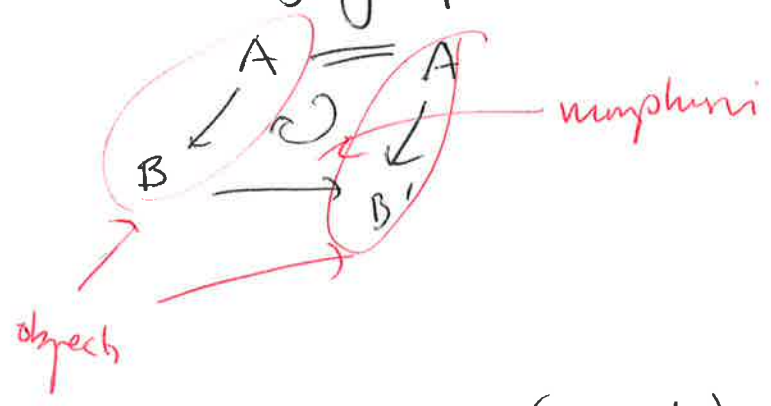
The cotangent complex

We will work at a convenient level of generality.

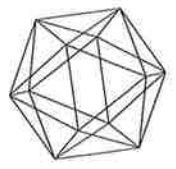
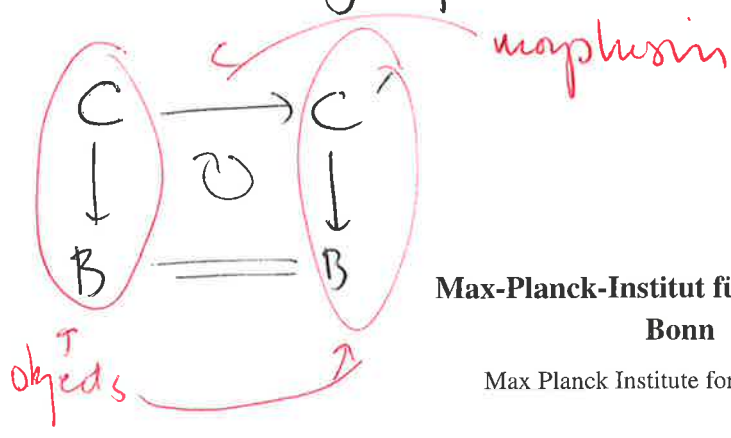
Fix k a characteristic zero field.

Let $\text{CAlg}_k^{\text{dg}}$ denote the (model) category of dg comm algebras over k .

For $A \in \text{CAlg}_k^{\text{dg}}$, let $\text{CAlg}_A^{\text{dg}}$ denote the (model) category of \checkmark^{com} dg algebras over A :



For $B \in \text{CAlg}_A^{\text{dg}}$, let $(\text{CAlg}_A^{\text{dg}})_B$ denote the (model) category of A -algebras sliced over B



Then Sq Ext: $\text{Mod}_B^{dg} \longrightarrow (\text{CAly}_A^{dg})/B$

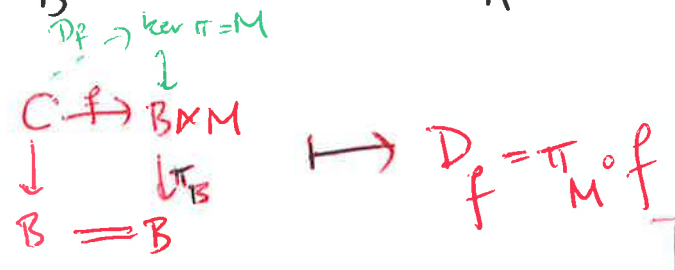
$$M \longmapsto \begin{pmatrix} B \otimes M \\ \text{Fl} \\ B \\ \text{Fl} \\ B \otimes M \end{pmatrix} \quad \begin{matrix} (b, m) \circ (b', m') \\ \text{"} \\ (bb', bm' + b'm) \end{matrix}$$

is well-defined.

Moreover

$$\text{CAly}_B^{dg}(C, B \otimes M) \cong \text{Der}_A^{dg}(C, M) \cong \text{Mod}_B^{dg}(B \otimes_C \Omega'_{C/A}, M)$$

universal property of Kähler differentials



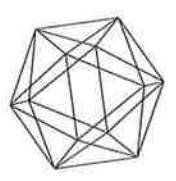
"indecomposables"

Recall: $\Omega'_{C/A} = J/J^2$, $J = \text{Ker}(C \otimes_A C \xrightarrow{m_C} C)$

$$\begin{array}{ccc} \text{Mod}_B^{dg}(B \otimes_C \Omega'_{C/A}, M) & \cong & \text{Mod}_C^{dg}(\Omega'_{C/A}, M) \ni g \\ \cong \downarrow & & \downarrow \\ \text{Der}_A^{dg}(C, M) & \ni & g \circ d_{dR} \end{array}$$

base change from C to B

$$\begin{array}{l} d_{dR}: C \rightarrow \Omega'_{C/A} \\ x \mapsto [x \otimes 1 - 1 \otimes x] \end{array}$$



Hence

Prop There is a Quillen adjunction $\text{check SqExt preserves fibrant \& acyclic fibrations}$

$$B \otimes \Omega_{-/A}^d : (\text{CAlg}_{/A}^{\text{dg}})_{/B} \rightleftarrows \text{Mod}_B^{\text{dg}} : \text{SqExt}$$

Def The cotangent complex $\mathbb{L}_{B/A}$ is any B -module quasiisomorphic to

$$B \otimes_P \Omega_{P/A}^1$$

where $P \xrightarrow{\sim} B$ is a cofibrant replacement for B in $\text{CAlg}_{/A}^{\text{dg}}$.

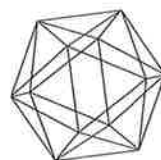
Trivial example

let $A=k$, $B=k[x_1, \dots, x_n] = \text{Sym}(\{x_1, \dots, x_n\})$

Then B is already cofibrant, since it's free

Hence
$$\mathbb{L}_{B/A} \simeq \Omega_{B/A}^1 = \bigoplus_{i=1}^n k[x_1, \dots, x_n] dx_i$$

~~Some notes for the...~~



Remark For more complicated B , one usually constructs a "semi-free resolution"

$$P = \left(k[x_\alpha]_{\alpha \in I}, \partial = \sum_{\alpha} f_\alpha \frac{\partial}{\partial x_\alpha} \right) \xrightarrow{\cong} B$$

$$|f_\alpha| - |x_\alpha| = 1$$

&

$$f_\alpha \in k[x_\beta]_{\beta < \alpha}, \text{ I ordered}$$

as $\Omega_{P/k}^1$ is easy to compute then. //

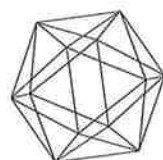
Lemma (Flat base change)

$$\begin{array}{ccc} \text{let } A & \xrightarrow{f} & A' \\ \phi \downarrow \text{P.O.} & & \downarrow \\ B & \longrightarrow & B' = A' \otimes_A B \end{array} \quad \begin{array}{l} \text{be a pushout diagram} \\ \text{in } \text{CAldg}_A \end{array}$$

where ϕ or f is flat. Then

$$B' \otimes_A \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{B'/A'}$$

is a quasi-iso of B' -modules.



Pf

Let $P \rightarrow B$ be a cofibrant replacement as an A -algebra.

Then

$$P' := A' \otimes_A P \longrightarrow A' \otimes_A B = B'$$

is a weak equivalence by the flatness assumption and P' is cofibrant as an A' -algebra.

Thus the natural map from base change

$$\begin{array}{ccc}
B' \otimes_B (B \otimes_P \Omega_{P/A}^1) & \longrightarrow & B' \otimes_{P'} \Omega_{P'/A'}^1 \\
\cong & & \cong \\
B' \otimes_B \mathbb{L}_{B/A} & & \mathbb{L}_{B'/A'}
\end{array}$$

is a weak equivalence. □

~~Similarly generalizing the right exact sequence in Kähler differentials:~~

~~...~~

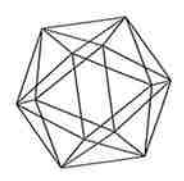
Similarly generalizing the right exact sequence in Kähler differentials:

Prop For $A \rightarrow B \rightarrow C$ maps in Catg , the natural maps

$$C \otimes_B \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{C/A} \longrightarrow \mathbb{L}_{C/B}$$

form a cofiber sequence.

"homotopy"



Application

$$\begin{array}{ccc}
 \text{CAlg}_k^{\text{aug}} = (\text{CAlg}_k)_k & \xrightarrow{\quad \perp \quad} & \text{Mod}_k \\
 & \xleftarrow{\text{Sq Ext}} & \\
 & \text{w/ the shift!} &
 \end{array}$$

~~L~~

where

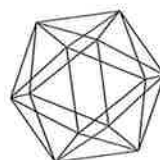
$$L(A) \simeq \left(k \otimes_A \mathbb{T}_{A/k} \right)[-1]$$

$$\simeq \mathbb{T}_{k/A}$$

use transitivity for $k \rightarrow A \rightarrow k$

Hence $-\Theta \circ \mathbb{D} \simeq \text{ID}_k \circ L : A \mapsto \mathbb{T}_{k/A}^{\vee}$

(Note that this is what you'd guess as a model for $\mathbb{T}_{k/A}$: The dual to cotangent!)



Now some useful results follow.

Lemma

For $g \in \text{Lie } k$ s.t.

(a) ~~$g^d \equiv 0$~~ $g^d \equiv 0$, $d \leq 0$

(b) $\dim_k (g^d) < \infty$, $\forall d$,

the unit map $g \rightarrow \mathcal{D}(C_{\text{lie}}^+(g))$ is an equivalence.

★ STATE & PROVE COROLLARY FIRST!

Pf It suffices to verify there is an equivalence on the underlying cochain complexes.

The map $\mathcal{O}(g) \rightarrow \mathcal{O}(\mathcal{D}(C_{\text{lie}}^+(g))) \simeq \mathbb{L}_{k/C_{\text{lie}}^+(g)}^v$

has a predual

$$\mathbb{L}_{C_{\text{lie}}^+(g)/k} \otimes_k C_{\text{lie}}^+(g) \longrightarrow g^v[-1]$$

We will prove this is an equivalence.



(13)

Recall that $C_{\text{lie}}^+(g)$ is $\prod_n \text{Sym}^n(g^{\vee[-1]})$ as a graded algebra.
 fix a basis x_1, \dots, x_n for $(g^+)^{\vee}$, so that

$$C_{\text{lie}}^+(g)^0 \cong k[x_1, \dots, x_n]$$

$$\text{Let } A = \left(\bigoplus_n \text{Sym}^n(g^{\vee[-1]}) \right)_{\mathfrak{a}} \xrightarrow{\phi} C_{\text{lie}}^+(g)$$

$$\text{and so } A^0 = k[x_1, \dots, x_n] \xrightarrow{\phi^0} k[x_1, \dots, x_n]$$

Note that by the hypotheses,

$$A \otimes_{k[x_1, \dots, x_n]} k[x_1, \dots, x_n] \xrightarrow{\cong} C_{\text{lie}}^+(g)$$

Indeed, the map

$$\begin{array}{ccc} k[x_1, \dots, x_n] & \longrightarrow & k[x_1, \dots, x_n] \\ \downarrow & & \downarrow \\ A & \longrightarrow & C_{\text{lie}}^+(g) \end{array}$$

is a pushout square in Alg_k as ϕ^0 is flat

$$\text{and so } H_{\mathfrak{a}}^n(A) \otimes_{k[\cdot]} k[\cdot] \xrightarrow{\cong} H_{\mathfrak{a}}^n(g)$$



Thus by flat base change:

$$\mathbb{L}_{C_{ie}^*(g)/A} \otimes_A k \simeq \mathbb{L}_{k[x_1, \dots, x_n]/k[x_1, \dots, x_n]} \otimes_{k[x_1, \dots, x_n]} k \simeq \mathbb{L}_{R/k} \simeq 0$$

where $R = k[x_1, \dots, x_n] \otimes_{k[\cdot]} k \simeq k$.

Now the predual map can be identified with

$$\mathbb{L}_{A/k} \otimes_A k \longrightarrow g^V[-1]$$

but A is cofibrant (as it is semifree) ^{so} use cotriple sequence

$$\mathbb{L}_{A/k} \otimes_A k \simeq g^V[-1] \quad \square$$

★ Cor For $n \geq 0$, $\text{Free}_{ie}(k[-n-1])$ satisfies the conditions (a) & (b) so

$$\mathbb{D}^{(n)} \xrightarrow{\simeq} \mathbb{D}(C_{ie}^*(g^{(n)}))$$

so $\mathbb{D}(k \oplus k^{(n)}) \simeq g^{(n)}$.

